

Virtual Braids and the L -Move

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Abstract

In this paper we prove a Markov theorem for virtual braids and for analogs of this structure including flat virtual braids and welded braids. The virtual braid group is the natural companion to the category of virtual knots, just as the Artin braid group is the natural companion to classical knots and links. In this paper we follow L -move methods to prove the Virtual Markov theorems. One benefit of this approach is a fully local algebraic formulation of the theorems in each category.

Keywords: classical knot, virtual knot, flat virtual knot, welded knot, Reidemeister moves, detour moves, forbidden moves, L -moves, virtual L -moves, stabilization moves, conjugation, exchange moves. (Mathematics Subject Classification 2000: 57M27)

1 Introduction

In this paper we prove a Markov theorem for the virtual braid group and for some analogs of this structure. The virtual braid group is the natural companion to the category of virtual knots, just as the Artin braid group is to classical knots and links. In classical knot theory the braid group gives a fundamental algebraic structure associated with knots. The Alexander theorem tells us that every knot or link can be isotoped to braid form. The capstone of this relationship is the Markov theorem, giving necessary and sufficient conditions for two braids to close to the same link (where sameness of two links means that they are ambient isotopic).

The Markov theorem in classical knot theory is not easy to prove. The theorem was originally stated by Markov with three moves and then Weinberg reduced them to the known two moves [29, 38]. The first complete proof is due to Birman [4]. Other published proofs are due to Bennequin [3], Morton [30], Traczyk [35] and Lambropoulou [24, 25]. In this paper we shall follow the “ L -Move” methods of Lambropoulou. In the L -move approach to the Markov theorem, one gives a very simple uniform move that can be applied anywhere in a braid to produce a braid with the same closure. This move, the L -move, consists in cutting a strand of the braid and taking the top of the cut to the bottom of the braid (entirely above or entirely below the braid) and taking the bottom of the cut to the top of the braid (uniformly above or below in

correspondence with the choice for the other end of the cut). See Fig. 15 for an illustration of a classical L -move. One then proves that two braids have the same closure if and only if they are related by a sequence of L -moves. Once this L -Move Markov theorem is established, one can reformulate the result in various ways, including the more algebraic classical Markov theorem that uses conjugation and stabilization moves to relate braids with equivalent closures.

Up to now [25, 10, 26] the L -moves were only used for proving analogues of the Markov theorem for classical knots and links in 3-manifolds (with or without boundary). Our approach to a Markov theorem for virtual knots and links follows a similar strategy to the classical case, but necessarily must take into account properties of virtual knots and links that diverge from the classical case. In particular, we use L -moves that are purely virtual, as well as considering the effect of allowed and forbidden moves of the virtual braids. The strategy for our project is to first give a specific algorithm for converting a virtual link diagram to a virtual braid. This algorithm is designed to be compatible with the L -moves. We prove that if two virtual diagrams are related to each other by a sequence of virtual isotopy moves, then the corresponding braids are related by virtual L -moves and real conjugation. The exact description of the L -moves for virtual braids is found in Section 2.2 (Definitions 2, 3, 4 and Figs. 11, 12, 13).

The L -Move Markov theorem for virtual braids is proved in Section 3 (see Theorem 2). Once the L -Move theorem is proved, it is a natural task to reformulate it in algebraic terms. In Section 4 we formulate and prove a local algebraic Markov theorem for virtual braids (Theorem 3). This and the L -Move Markov theorem for virtuals are the key results of our paper. In Section 5 we recover the Markov theorem for virtual braids proved by Kamada in [15]. Such theorems are important for understanding the structure and classification of virtual knots and links. The L -move approach provides a flexible conceptual center from which to deduce many results. In particular, it would surely be quite difficult to compare our local algebraic formulation of the Markov theorem with that of Kamada without the fundamental L -move context. Our local algebraic version of the Markov theorem promises to be useful for constructing new invariants of virtual knots and links.

We conclude the paper with descriptions of variations of our Markov theorem for other categories of braids, such as flat virtual braids, welded braids and virtual unrestricted braids, in Section 6 (see Theorems 4, 5, 6, 7, 8). For the case of welded braids our results coincide with the results of Kamada [15]. Finally, in Section 7 we describe the general pattern for obtaining quantum invariants via solutions to the Yang-Baxter equation and Hecke algebra type invariants of virtual links via braids. These topics will be the subject of our future research.

2 Virtual Knot Theory

Virtual knot theory is an extension of classical diagrammatic knot theory. In this extension one adds a *virtual crossing* (see Fig. 1) that is neither an over-crossing nor an under-crossing. A virtual crossing is represented by two crossing arcs with a small circle placed around the crossing point.

Virtual diagrams can be regarded as representatives for oriented Gauss codes (Gauss diagrams) [18, 9]. Some Gauss codes have planar realizations, and these correspond to classical knot diagrams. Some codes do not have planar realizations. An attempt to embed such a code in the plane leads to the production of virtual crossings.

Another useful topological interpretation for virtual knot theory is in terms of embeddings of links in thickened surfaces, taken up to addition and subtraction of empty handles. Regard each virtual crossing as a shorthand for a detour of one of the arcs in the crossing through a 1-handle that has been attached to the 2-sphere of the original diagram (see Fig. 1). By interpreting each virtual crossing in this way, we obtain an embedding of a collection of circles into a thickened surface of genus the number of virtual crossings in the original diagram. See [6, 14, 18, 20]. We say that two such surface embeddings are *stably equivalent* if one can be obtained from another by isotopy in the thickened surfaces, homeomorphisms of the surfaces and the addition or subtraction of empty handles. Then we have the following theorem [20, 6]: *Two virtual link diagrams are isotopic if and only if their correspondent surface embeddings are stably equivalent.*

A third way to make a topological interpretation of virtual knots and links is to form a ribbon-neighborhood surface (sometimes called an *abstract link diagram* [14]) for a given virtual knot or link, as illustrated in Fig. 1. In this figure we show how a virtual trefoil knot (two classical and one virtual crossing) has the classical crossings represented as diagrammatic crossings in disks, which are connected by ribbons, while the virtual crossing is represented by ribbons that pass over one another without interacting. The abstract link diagram is shown embedded in three dimensional space, but it is to be regarded without any particular embedding of the surface. Thus it can be represented with the ribbons for the virtual crossings switched. These abstract link diagrams give the least surface embedding (with boundary) that can represent a given virtual link diagram.

Isotopy moves on virtual diagrams generalize the ordinary Reidemeister moves for classical knot and link diagrams. See Fig. 2, where all variants of the moves should be considered. In this work, virtual diagrams are always oriented, so the isotopy moves will be considered with all possible choices of orientations. One can summarize the moves on virtual diagrams as follows: The real crossings interact with one another according to the *classical Reidemeister moves* (Part A of Fig. 2). Virtual crossings interact with one another by *virtual Reidemeister moves* (Part B of Fig. 2). The key move between virtual and classical crossings is shown in Part C of Fig. 2. Here a consecutive sequence of

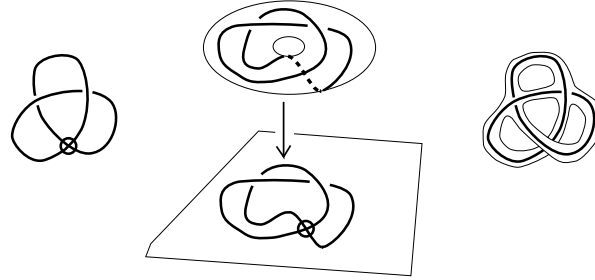


Figure 1: A virtual trefoil and its surface realizations.

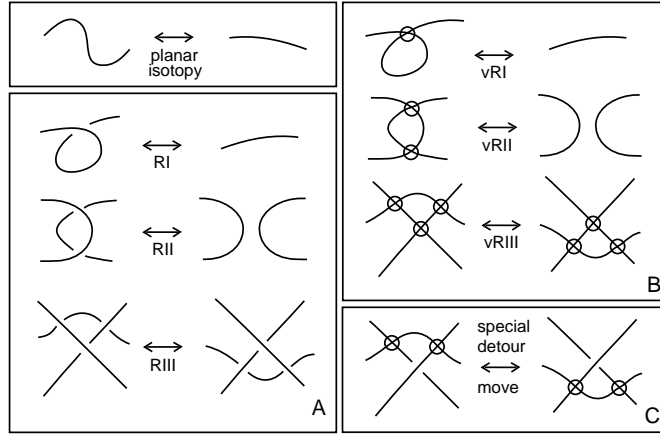


Figure 2: Reidemeister moves for virtuals.

two virtual crossings can be moved across a single classical crossing. We will call it a *special detour move*, because it is a special case of the more general *detour move* indicated in Fig. 3. All these moves together with the *planar isotopy moves* (top left of Fig. 2) generate an equivalence relation in the set of virtual knot and link diagrams, called *virtual equivalence* or *virtual isotopy*.

In the detour move, an arc in the diagram that contains a consecutive sequence of virtual crossings can be excised, and the arc re-drawn, transversal to the rest of the diagram (or itself), adding virtual crossings whenever intersections occur. See Fig. 3. In fact, each of the moves in Parts B and C of Fig. 2 can be regarded as special cases of the detour move. By similar arguments as in the classical Reidemeister theorem, it follows that any detour move can be achieved by a finite sequence of local steps, each one being a Reidemeister move from Part B or C. A succinct description of virtual isotopy is that it is generated by classical Reidemeister moves *and* the detour move.

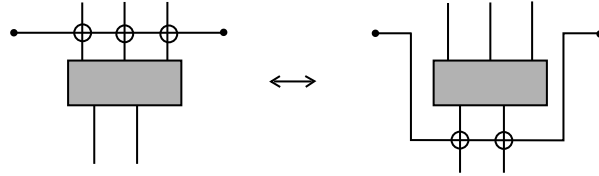


Figure 3: **The detour move.**

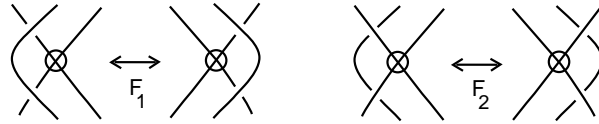


Figure 4: **The forbidden moves.**

We note that a move analogous to a special detour move but with two real crossings and one virtual crossing is a *forbidden move* in virtual knot theory. There are two types of forbidden moves: One with an over arc, denoted F_1 , and another with an under arc, denoted F_2 . See [18] for explanations and interpretations. Variants of the forbidden moves are illustrated in Fig. 4.

We know [18, 9] that classical knot theory embeds faithfully in virtual knot theory. That is, if two classical knots are equivalent through moves using virtual crossings, then they are equivalent as classical knots via standard Reidemeister moves. With this approach, one can generalize many structures in classical knot theory to the virtual domain, and use the virtual knots to test the limits of classical problems, such as the question whether the Jones polynomial detects knots. Counter-examples to this conjecture exist in the virtual domain. It is an open problem whether some of these counter-examples are equivalent to classical knots and links.

3 Virtual Braids

Just as classical knots and links can be represented by the closures of braids, so can virtual knots and links be represented by the closures of virtual braids [19, 15, 21]. A *virtual braid* on n strands is a braid on n strands in the classical sense, which may also contain virtual crossings. The closure of a virtual braid is formed by joining by simple arcs the corresponding endpoints of the braid on its plane. Like virtual diagrams, a virtual braid can be embedded in a ribbon surface. See Fig. 5 for an example.

The set of isotopy classes of virtual braids on n strands forms a group, the virtual braid group, denoted VB_n , that can be described by generators and

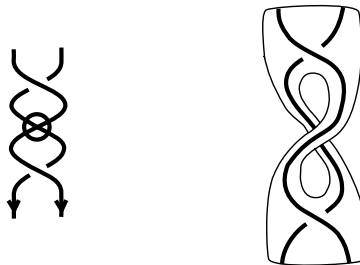


Figure 5: A virtual braid and its ribbon surface realization.

relations, generalizing the generators and relations of the classical braid group [19]. This structure of virtual braids is worth study for its own sake. The virtual braid group is an extension of the classical braid group by the symmetric group. See [18], [2], [21]. It is worth remarking that classical braids embed in virtual braids just as classical links embed in virtual links. This fact may be most easily deduced from [23].

Virtual braids representing isotopic virtual links are related via a Markov-type virtual analogue. In [15] Kamada proves a Markov theorem for virtual braids, giving a set of moves on virtual braids that generate the same equivalence classes as the virtual link types of their closures. For reference to previous work on virtual links and braids the reader should consult [2, 6, 7, 9, 11, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 27, 28, 32, 36, 37].

3.1 Braiding Virtual Diagrams

It is easily seen that the classical Alexander theorem [1, 5] generalizes to virtuals.

Theorem 1. *Every (oriented) virtual link can be represented by a virtual braid, whose closure is isotopic to the original link.*

Indeed, it is quite easy to braid a virtual diagram. In [21] we gave, for example, a new braiding algorithm, which is applicable, in fact, to all the categories in which braids are constructed. The idea of that algorithm is very similar to the braiding algorithm of Kamada [15], and it is the following: we consider a virtual link diagram arranged in general position with respect to the height function. We then rotate all crossings of the diagram on the plane, so that all arcs in the crossings are oriented downwards. We leave the down-arcs in place and eliminate the up-arcs, producing instead braid strands. The elimination of an up-arc is described in Fig. 9.

For the purposes of this paper, where we need to analyze how the isotopy moves on diagrams affect the final braids, we follow a different braiding process.

Figure 6: **The swing moves.**

Preparation for braiding. Firstly, for simplicity and without loss of generality, virtual link diagrams are assumed piecewise linear. Working in the piecewise linear category gives rise to another ‘move’: the *subdivision* of an arc into two smaller arcs, by marking it with a point. The vertices and the local maxima and minima are subdividing points of a diagram. Subdivision of an arc with no crossings can be regarded as a degenerate case of the planar isotopy move.

Furthermore, virtual link diagrams lie on the plane, which is equipped with the top-to-bottom direction. This makes our set-up liable to certain conventions. For example, an oriented virtual diagram contains only *up-arcs* and *down-arcs* (no horizontal arcs). It contains no horizontally aligned crossings, so as to have the crossings in the corresponding braid lying on different horizontal levels. Vertically aligned crossings or subdividing points are also not permitted, so as to avoid triple points when creating new strands or pairs of braid strands with the same endpoints. The above discussion gives rise to the following definition.

Definition 1. A virtual link diagram is said to be in *general position* if it does not contain any horizontal arcs and no two subdividing points or crossings are vertically or horizontally aligned, nor is a crossing coincident with a maximum or a minimum.

Clearly, any virtual diagram can assume general position by very small planar shifts. Note that, the arcs or points or crossings that violate Definition 1 may not be close in the diagram. For example, two aligned subdividing points may lie far away. The point is that the correcting shifts can be applied on only one of them, so, in this sense these shifts can be assumed local. Moreover, when bringing a virtual diagram to general position we meet certain choices. For example, in a parallel occurrence of a maximum and a minimum, either one can occur first in the vertical order. Different choices amount to local shifts of crossings and subdividing points with respect to the horizontal or the vertical direction. These local shifts shall be called *direction sensitive moves*.

The most interesting instances of such moves are the *swing moves*. See Fig. 6. A swing move avoids the coincidence of a maximum or minimum and a crossing, real or virtual. It turns out that adding the swing moves to our list of virtual isotopy moves makes redundant certain instances of Reidemeister moves involving horizontal arcs. For example, it is easily verified that an RII move with two horizontal arcs can be produced by an RII move with two vertical arcs, two swing moves and changes of relative positions of vertices.

It follows now easily that any two virtual diagrams in general position that

correspond to isotopic virtual diagrams will differ by the above direction sensitive moves and the Reidemeister moves for virtuals, all in general position. From now on, all diagrams will be assumed in general position.

The braiding. We are now ready to describe our braiding algorithm. The down-arcs will stay in place while the up-arcs shall be eliminated. Now, an up-arc will either be an arc of a crossing or it will be a “free up-arc”. We place each crossing containing one or two up-arcs in a small rectangular box with diagonals the arcs of the crossing, the *crossing box*. A crossing box is assumed sufficiently narrow, so that the vertical zone it defines does not intersect the zone of another crossing. The *free up-arcs* are arcs joining the crossing boxes. We first braid the crossings containing an up-arc, one by one, according to the crossing charts of Fig. 7. Except for the local crossings shown in the illustrations, all other crossings of the new braid strands with the rest of the diagram are virtual. This is indicated abstractly by placing virtual crossings at the ends of the new strands. The result is a virtual tangle diagram.

It is easy to verify that closing the corresponding braid strands of a braided crossing results in a virtual tangle diagram isotopic to the starting one. In Fig. 8 we illustrate this isotopy for one of the less obvious cases.

It remains to braid the free up-arcs. We braid a free up-arc by sliding it first across the right-angled triangle with hypotenuse the up-arc and with the right angle lying below it, so that *it crosses virtually* any other arcs of the original diagram that intersect the sliding triangle. A grey curved arc is illustrated to this effect in Fig. 9. We then cut the vertical segment at a point and we pull the two ends, the upper upward and the lower downward, so that the horizontal arc slopes slightly downwards, keeping them aligned, so that the two new braid strands cross any other part of the diagram *only virtually*. This is indicated in the illustrations by the virtual crossings on the final braid strands. We also care that the horizontal arc slopes slightly downwards, so that there is no conflict with not permitting horizontal arcs in Definition 1. Note that the prior elimination of crossings may cause vertical strands to cross virtually the free up-arc. This is not an obstacle for braiding it, since –by the detour move– the arc can slide virtually across these strands (see grey strands in Fig. 9).

In the end we created a pair of corresponding braid strands and we have one up-arc less. Note that joining the two corresponding braid strands yields a virtual tangle diagram obviously isotopic to the starting one, since from the free up-arc we created a stretched loop around the braid axis, which is detour isotopic to the arc. The braiding of a free up-arc is a *basic braiding move*.

After completing all braidings we obtain an open virtual braid, the closure of which is an oriented virtual link diagram isotopic to the original one. The braiding algorithm given above will braid any virtual diagram and, thus, it proves Theorem 1. \square

Remark 1. Because of the narrow zone condition for the crossings (see the beginning of the braiding discussion) the braidings of the crossings are independent, so their order is irrelevant. Moreover, because of the braid detour move,

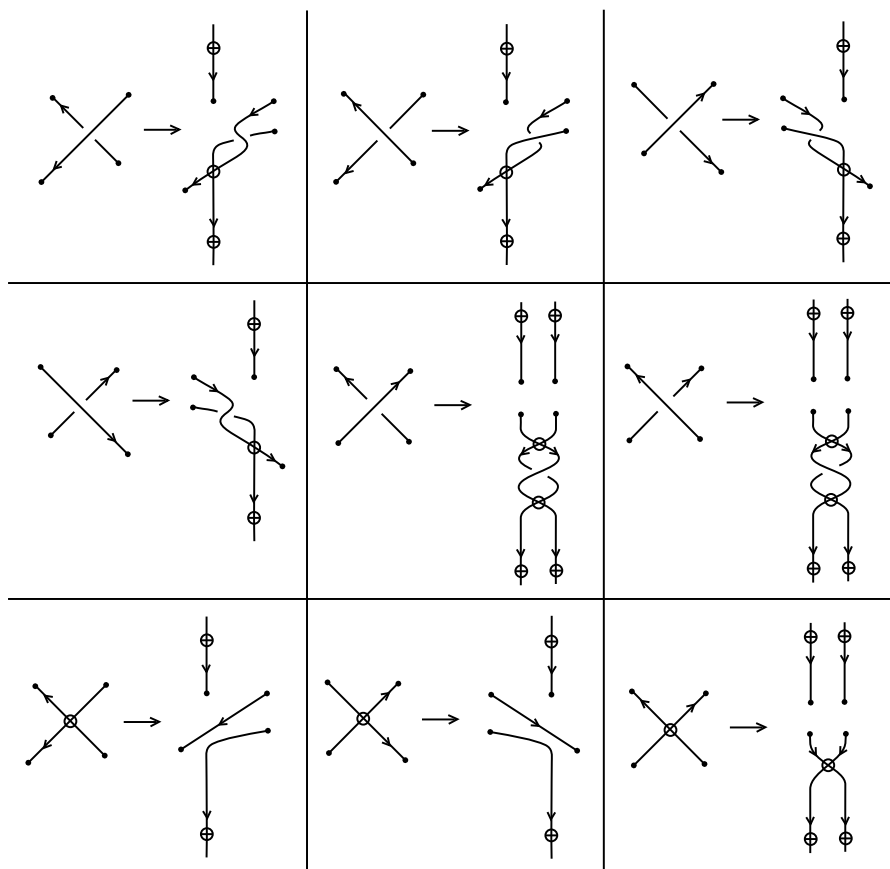


Figure 7: The braiding chart for crossings.

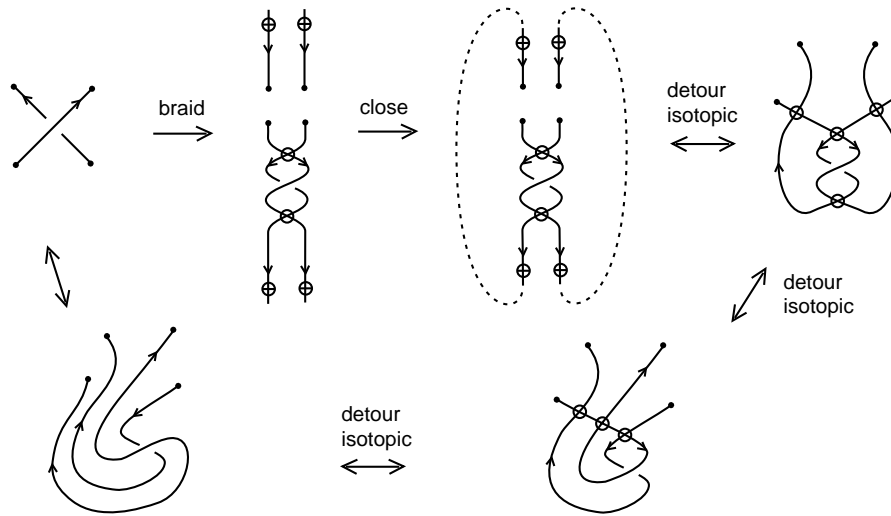


Figure 8: The closure of the braiding of a crossing.

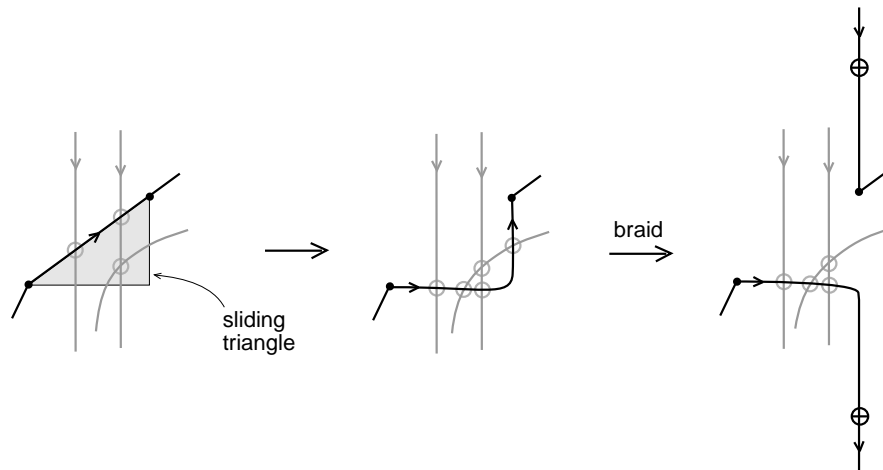
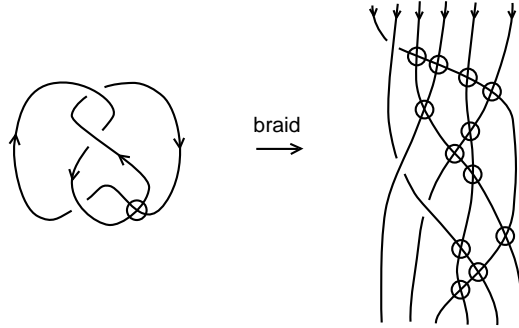


Figure 9: The basic braiding move.

Figure 10: **An example of braiding.**

it does not make any difference in which order we braid the free up-arcs. In fact, we could even braid any number of them before completing the braidings of the crossings.

Remark 2. The braidings of the crossings are also based on the basic braiding move. Using this, it is easy to verify that, if in the instances of the braiding chart we replace each arc by a number of parallel arcs with the same orientation and the same crossings, the resulting braids are L -equivalent to the ones we would obtain if we braid one by one the single crossings in the formation, according to the chart. This remark can save us from creating unnecessary extra braid strands.

The set-up of our virtual braiding resembles the one in [25] for classical links, but only to the extent that we consider piecewise linear diagrams on the plane, which is equipped with the top-to-bottom direction, and that the basic braiding move looks similar. With the forbidden moves in the theory, the choices needed here are completely different from the ones made in the classical set-up. For example, we are forced to braid a crossing of two up-arcs as one entity, not its arcs one by one. (Braiding crossings as rigid entities can, obviously, be applied also in the classical set-up for braiding knots and links.) In the classical set-up, braiding an over up-arc corresponds to pulling the new pair of braid strands over the rest of the diagram. Here it has to be always virtually. Another technical difference is that, in the classical set-up it was important to ensure that the sliding triangles have no intersections with other parts of the diagram. Here this assumption is not needed.

3.2 The L -equivalence for Virtual Braids

As in classical knot theory, the next consideration after the braiding is to characterize virtual braids that induce, via closure, isotopic virtual links. In this section we describe an equivalence relation between virtual braids, the L_v -equivalence. For this purpose we need to recall and generalize to the virtual

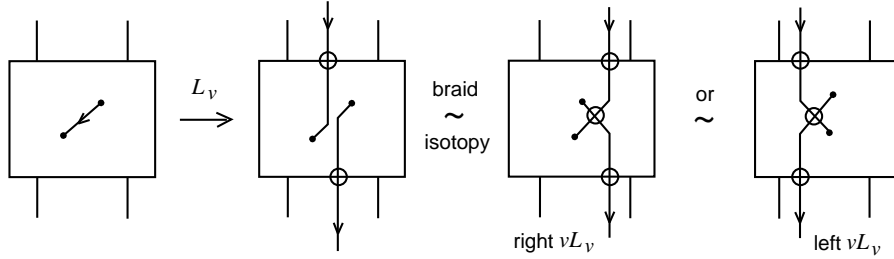


Figure 11: A Basic L_v -move and the two vL_v -moves.

setting the L -moves between braids to *virtual L -moves*, abbreviated to L_v -moves. The L -move (see Definition 5) was introduced in [24, 25], where it was used among other things to prove the ‘one-move Markov theorem’ for classical oriented links (cf. [25], Theorem 2.3), replacing the two well-known moves of the Markov equivalence: the *stabilization* that introduces a crossing at the bottom right of a braid and *conjugation* that conjugates a braid by a crossing.

Definition 2. A *basic L_v -move* on a virtual braid, consists in cutting an arc of the braid open and pulling the upper cutpoint downward and the lower upward, so as to create a new pair of braid strands with corresponding endpoints (on the vertical line of the cutpoint), and such that both strands cross entirely *virtually* with the rest of the braid. (In abstract illustrations this is indicated by placing virtual crossings on the border of the braid box.)

By a small braid isotopy that does not change the relative positions of endpoints, a basic L_v -move can be equivalently seen as introducing an in-box virtual crossing to a virtual braid, which faces either the *right* or the *left* side of the braid. If we want to emphasize the existence of the virtual crossing, we will say *virtual L_v -move*, abbreviated to vL_v -move. In Fig. 11 we give abstract illustrations. See also Fig. 16 for a concrete example.

Note that in the closure of a basic L_v -move or a vL_v -move the detoured loop contracts to a kink. This kink could also be created by a real crossing, positive or negative. So we define:

Definition 3. A *real L_v -move*, abbreviated to $+L_v$ -move or $-L_v$ -move, is a virtual L -move that introduces a real in-box crossing (positive or negative) on a virtual braid, and it can face either the *right* or the *left* side of the braid. See Fig. 12 for abstract illustrations.

If the crossing of the kink is virtual, then, in the presence of the forbidden moves, there is another possibility for an L_v -move on the braid level. Namely, we have:

Definition 4. A *threaded L_v -move* on a virtual braid is a virtual L -move with a virtual crossing in which, before pulling open the little up-arc of the kink, we

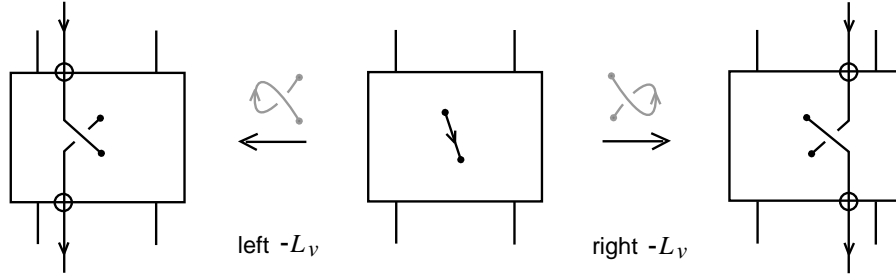


Figure 12: **Left and right real L_v -moves.**

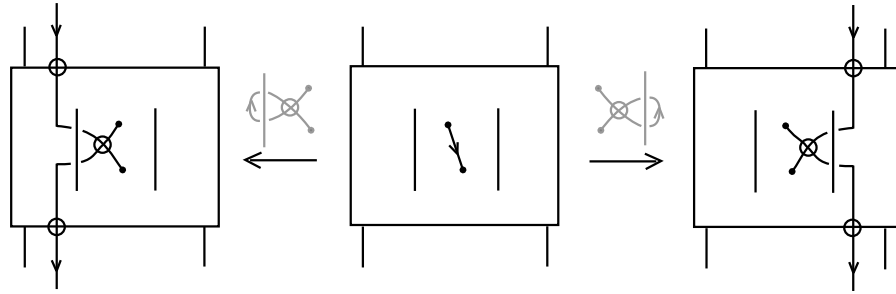


Figure 13: **Left and right under-threaded L_v -moves.**

perform a Reidemeister II move with real crossings, using another arc of the braid, the *thread*. See Fig. 13. There are two possibilities: an *over-threaded L_v -move* and an *under-threaded L_v -move*, depending on whether we pull the kink over or under the thread, both with the variants *right* and *left*.

Note that a threaded L_v -move cannot be simplified in the braid. If the crossing of the kink were real, then, using a braid RIII move with the thread, the move would reduce to a real L_v -move. Similarly, if the forbidden moves were allowed, a threaded L_v -move would reduce to a vL_v -move.

Remark 3. As with a braiding move, the effect of a virtual L -move, basic, real or threaded, is to stretch (and cut open) an arc of the braid around the braid axis using the detour move, after twisting it and possibly after threading it. Conversely, such a move between virtual braids gives rise to isotopic closures, since the virtual L -moves shrink locally to kinks (grey diagrams in Figs. 12 and 13).

Conceivably, the ‘threading’ of a virtual L -move could involve a sequence of threads and Reidemeister II moves with over, under or virtual crossings, as Fig. 14 suggests. The presence of the forbidden moves does not allow for

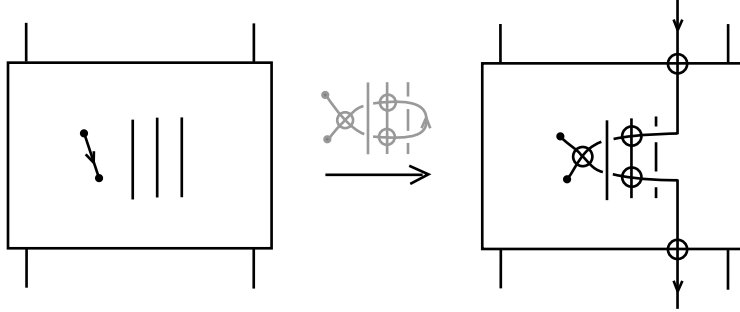


Figure 14: A right multi-threaded L_v -move.

simplifications on the braid level. We show later that such *multi-threaded L_v -moves* follow from the threaded L_v -moves, up to real conjugation.

We finally introduce the notion of a classical L -move, adapted to our set-up.

Definition 5. A *classical L_{over} -move* resp. *L_{under} -move* on a virtual braid consists in cutting an arc of the virtual braid open and pulling the two ends, so as to create a new pair of braid strands, which run both *entirely over* resp. *entirely under* the rest of the braid, and such that the closures of the virtual braids before and after the move are isotopic. See Fig. 15 for abstract illustrations. A classical L -move (over or under) may also introduce an in-box crossing, which may be positive, negative or virtual, or it may even involve a thread.

In order that a classical L -move between virtual braids is *allowed*, in the sense that it gives rise to isotopic virtual links upon closure, it is required that the virtual braid has no virtual crossings on the entire vertical zone either to the left or to the right of the new strands of the L -move. We then perform the isotopy on the side with no virtual crossings. We show later that the allowed L -moves can be expressed in terms of L_v -moves and real conjugation. It was the classical L -moves that were introduced in [24, 25], and they replaced the two equivalence moves of the classical Markov theorem. Clearly, in the classical set-up these moves are always allowed, while the presence of forbidden moves can preclude them in the virtual setting.

In Fig. 16 we illustrate an example of various types of L -moves taking place at the same point of a virtual braid.

4 The L -move Markov Theorem for Virtual Braids

It is clear that different choices when applying the braiding algorithm as well as local isotopy changes on the diagram level may result in different virtual braids. In this section we show that *real conjugation* (that is, conjugation by a real crossing) and some variations of the L_v -moves (recall Definitions 2, 3, 4)

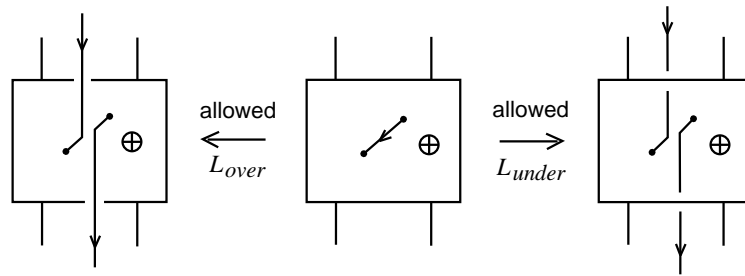


Figure 15: The allowed classical L -moves.

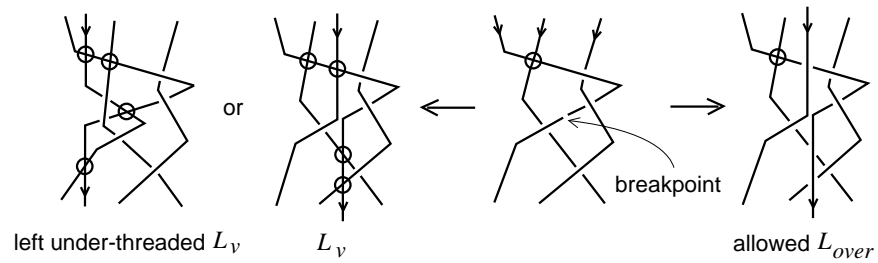


Figure 16: A concrete example of introducing L -moves.

capture and reflect on the braid level all instances of isotopy between virtual links.

Theorem 2 (L -move Markov theorem for virtuals). *Two oriented virtual links are isotopic if and only if any two corresponding virtual braids differ by virtual braid isotopy and a finite sequence of the following moves or their inverses:*

- (i) *Real conjugation.*
- (ii) *Right virtual L_v -moves.*
- (iii) *Right real L_v -moves.*
- (iv) *Right and left under-threaded L_v -moves.*

Definition 6. Moves (i)–(iv) together with virtual braid isotopy generate an equivalence relation in the set of virtual braids, the L -equivalence.

Note that in the statement of Theorem 2 we do not use virtual conjugation, basic L_v -moves, left virtual or real L_v -moves, allowed classical L -moves, over-threaded L_v -moves (right or left) and multi-threaded L_v -moves. In the next lemmas we show that all these moves (except for the left real L_v -moves) follow from the L -equivalence. We shall then use them freely in the proof of Theorem 2. The proof that left real L_v -moves follow from the L -equivalence shall be given at the end of the proof of the theorem (Lemma 9).

Lemma 1. *Virtual conjugation can be realized by a sequence of basic and virtual L_v -moves.*

Proof. The proof is an adaptation for virtual conjugation of a similar proof of R. Häring-Oldenburg for classical braids and real conjugation [10]. In Fig. 17 we start with a virtual braid conjugated by a virtual crossing. After performing an appropriate basic L_v -move and braid isotopy, and after undoing another virtual L_v -move we end up with the original braid. \square

Note that the ‘trick’ of Fig. 17 would not work in the case of real conjugation. In fact, we conjecture that real conjugation cannot be generated by virtual L -moves.

Lemma 2. *Basic and left virtual L_v -moves follow from right virtual L_v -moves and braid isotopy.*

Proof. The proof is illustrated in Fig. 18. \square

It is easy to see that an allowed classical L -move reduces, up to real conjugation and classical braid relations, to a right or left real L_v -move at the extreme right or left of the braid box. See the discussion after Remark 7.

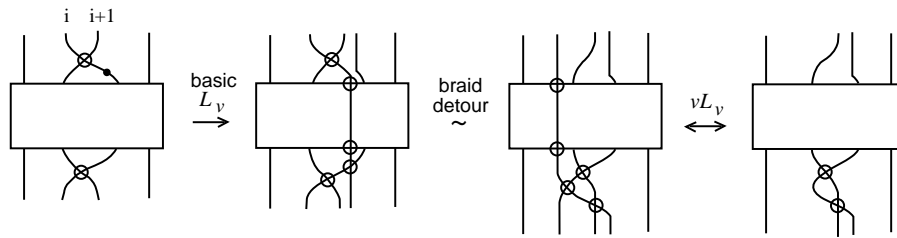


Figure 17: Conjugation by a virtual crossing is a composition of L_v -moves.

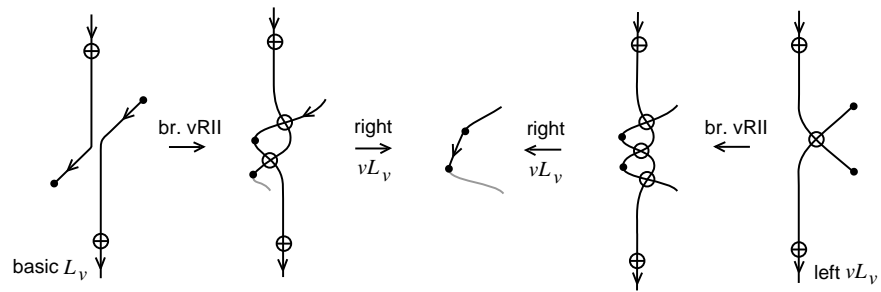


Figure 18: Basic and left virtual L_v -moves as right vL_v -moves.

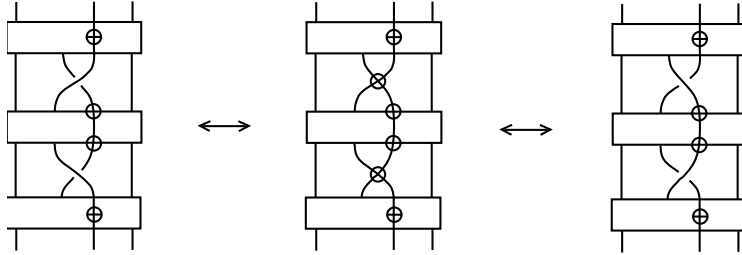


Figure 19: **In-box exchange moves.**

We shall now prove a key lemma about ‘in-box exchange moves’.

Definition 7. An *in-box exchange move* is a move between virtual braids as illustrated in Fig. 19 between the first two or the last two pictures, together with the two variants with facing the left (obtained by reflecting the diagrams in a vertical axis).

Lemma 3. *The in-box exchange moves follow from L_v -moves and real conjugation.*

Proof. The proof is illustrated in Fig. 20 for one variant of the in-box exchange moves. For the second step we point out that the real conjugation we do here can be carried out just above and below the middle box, for the following reason: since the i th and $(i+1)$ st strands cross the top and bottom braid boxes virtually, any crossing in their vertical zone can be braid detoured away. So, this vertical zone will only contain parts of other strands (drawn in grey), crossing the i th and $(i+1)$ st strand virtually. Then, the real crossing formed by these two strands can be braid detoured to the top, get real conjugated to the bottom and then pass in the same manner to the region above the bottom box. \square

The next lemma shows that in the L -equivalence we only need, indeed, one type of threaded L_v -moves, say the under-threaded (left and right), recall Fig. 13.

Lemma 4. *The over-threaded L_v -moves follow from the L -equivalence moves of Definition 6.*

Proof. Lemma 3 is the key. Indeed, as illustrated in Fig. 21, a right over-threaded L_v -move gives rise to an in-box exchange move of the same type as the one in Fig. 20. So, applying Lemma 3 involves only L -equivalence moves. Similarly, the in-box exchange move facing the left (with a top negative real crossing) involves only a virtual L_v -move and a left under-threaded L_v -move. Thus, a left over-threaded L_v -move follows also from L -equivalence moves. \square

As a result of Lemmas 3 and 4 and their proofs we have the following.

Corollary 1. *The in-box exchange moves follow from the L -equivalence moves.*

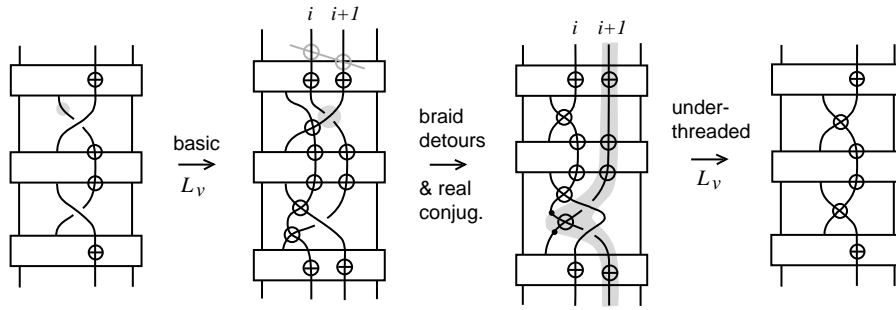


Figure 20: An in-box exchange move via L_v -moves and real conjugation.

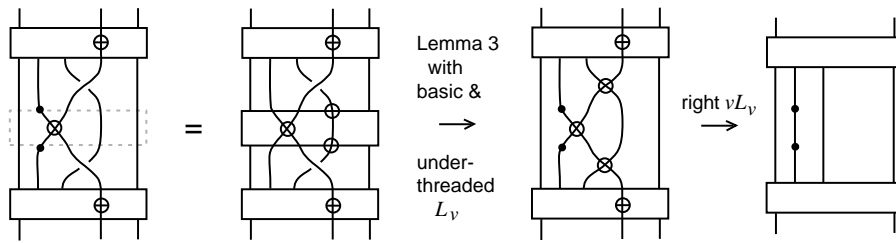


Figure 21: An over-threaded L_v -move via the L -equivalence moves.

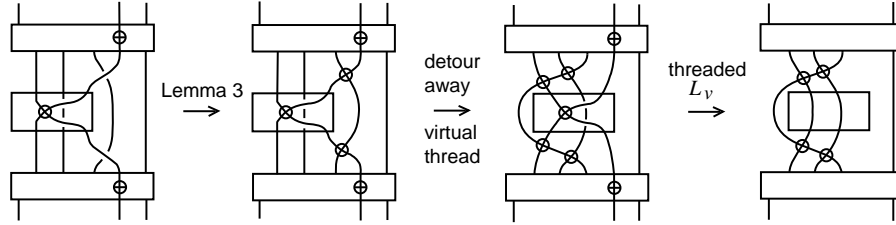


Figure 22: A multi-threaded L_v -move follows from the L -equivalence moves.

Proof. Indeed, when in the proof of Lemma 3 we reach an over-threaded L_v -move, we apply the process in Fig. 20 with just one virtual crossing in the middle box. \square

Lemma 5. *The multi-threaded L_v -moves are consequences of the L -equivalence moves.*

Proof. Notice first that all threads can be assumed real, as virtual threads can be braid-detoured away around the virtual crossing of the move. In Fig. 22 we illustrate the last step of the proof. We assume any number of real threads inside the middle braid box, instead of just one illustrated here. Then, by Lemma 3 we exchange the two real crossings of the outer thread with two virtual ones and we braid-detour away the virtual thread. We proceed like this until we are left with one real thread. If in the application of Lemma 3 or in the last step an over-threaded L_v -move is created, apply Lemma 4 and Corollary 1. \square

Remark 4. The in-box exchange moves of Definition 7 generalize the virtual exchange moves defined by Kamada in [15], which he used in formulating and proving a Markov type theorem for virtual braids. See Section 5 for details.

Proof of Theorem 2. Clearly, L -equivalent braids have isotopic closures. We have to show the converse.

The proof splits into two parts: the *technical part* and the *isotopy part*. In the technical part we compare virtual braids resulting from different choices made on a given virtual diagram during the braiding process. The isotopy part consists in comparing virtual braids corresponding to virtual diagrams that are related either by different choices made when bringing a diagram to general position (recall Definition 1) or by the virtual isotopy moves.

We first discuss the technical part. Since our braiding is quite rigid, the only choices made during the braiding process are the subdividing points and the order of the braiding moves. The order of the braiding moves is irrelevant, according to Remark 1. Subdividing points are needed for marking the up-arcs and the crossing boxes. Assume now that our diagram is equipped with a choice of subdividing points. In order to compare it to a different choice of subdividing points we need the following lemma.

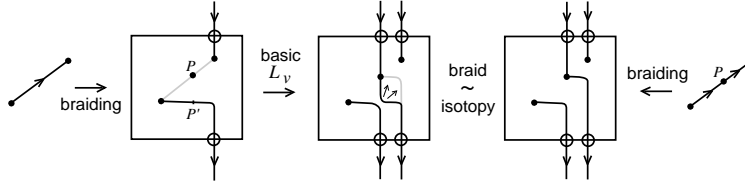


Figure 23: The proof of Lemma 6.

Lemma 6. *If we add to an up-arc an extra subdividing point, the corresponding braids differ by basic L_v -moves.*

Proof. Assume first that the up-arc is a free up-arc. Without loss of generality we have eliminated all other free up-arc and crossings containing up-arc of the diagram except for the up-arc in question and its subdivided replacement. We complete the braiding by eliminating the up-arc. In Fig. 23 we let P be the new subdividing point of the up-arc and P' its projection on the horizontal arc (slightly sloping downwards) created by the braiding. We perform a basic L_v -move at P' and, by a small braid planar isotopy, we obtain the braid that would result from the original diagram with the subdividing point P included.

If, now, the up-arc is an arc inside a crossing box, then we create a similar smaller box inside the original, using the new subdividing point, and we complete the braiding of the new formation. Again, we will find that the corresponding braids differ by two or four basic L_v -moves, depending on whether the crossing contains one or two up-arc. \square

Corollary 2. *Given any two subdivisions S_1 and S_2 of a virtual diagram, the corresponding braids are L -equivalent.*

Indeed, consider the subdivision $S_1 \cup S_2$, which is a common refinement of S_1 and S_2 , and apply repeatedly Lemma 6 to S_1 and to S_2 .

We proceed now with the isotopy part of the proof of Theorem 2. The choices we have when bringing a virtual diagram to general position are related to the direction sensitive moves (recall discussion after Definition 1). These, as well as the virtual isotopy moves, are all local. Thus, given two virtual diagrams that differ by such a move, we may assume that both have been braided everywhere, except for the arcs and crossings inside the regions of the local move. After completing the braiding, we compare two virtual braids, which are identical except for the effect of the move on each. In the figures that follow we focus only on the local moves and their braidings, dropping the abstract box.

Lemma 7. *Virtual diagrams in general position that differ by direction sensitive moves correspond to virtual braids that differ by basic and virtual L_v -moves.*

Proof. Repairing a horizontal arc corresponds to a planar isotopy move. If the arc in the move is an up-arc, the move boils down to subdivision of an up-arc

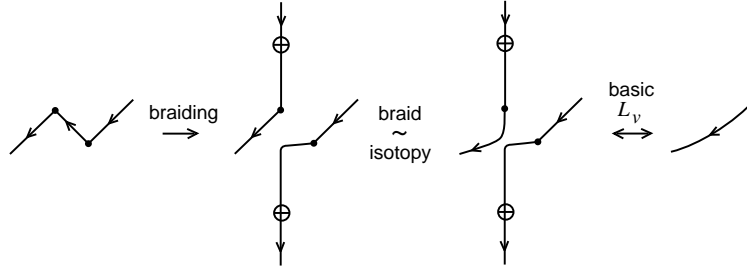


Figure 24: **Checking one case of planar isotopy.**

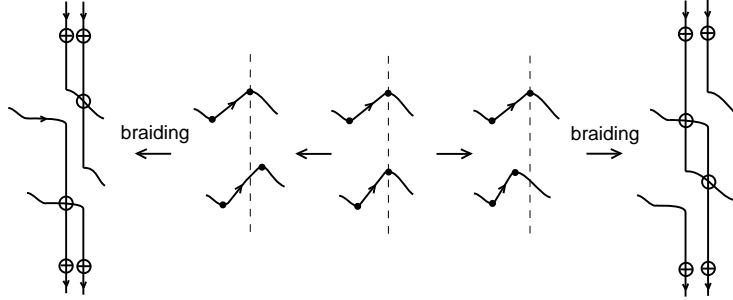


Figure 25: **An instance of vertical alignment.**

(Lemma 6), basically because subdivision can be seen as a degenerate case of planar isotopy. (We refer the reader to [25] for details.) In Fig. 24 we check planar isotopy in the case of a down-arc.

Changes of relative heights of crossings or subdividing points yield – up to virtual braid relations – the same virtual braids. Also, vertical alignment of crossings or subdividing points can be repaired by local sidewise shifts. In Fig. 25 we illustrate a case of vertical alignment of two subdividing points and its braided resolutions. We only show the two up-arcs containing the subdividing points (everything else is already braided), the alignment of which is indicated by a dotted line. Note that, up to a braid vR_{II} move, the two braids are conjugates by a virtual crossing. All other cases of vertical alignment are based on the same idea, possibly involving conjugation by more than one virtual crossing.

We shall now check the swing moves. There are various cases, depending on the orientation, the type of crossing and the minimum/maximum. The ones with a virtual crossing are very easy to check. In Figs. 26 and 27 we check two cases with a real crossing.

In many parts of Fig. 27 we have drawn in grey the continuation of an arc. In the last instance this is needed for comparing the final braids of the two sides

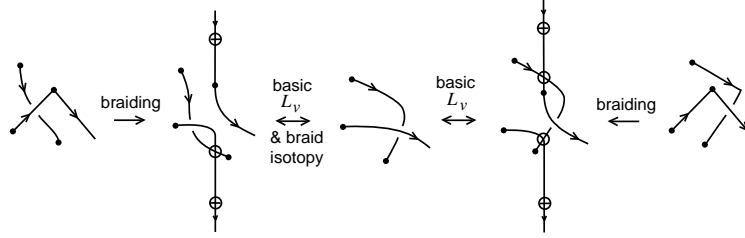


Figure 26: The first case of the swing moves.

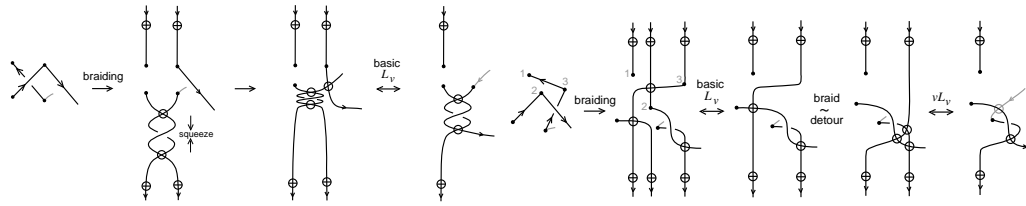


Figure 27: The second case of the swing moves.

of the move. The key point here is that this grey arc is part of the braiding of an up-arc, so its crossing with our braid diagram will be virtual. Recall our assumption, that in the regions of the local moves there are no other crossings of the original diagram. Note, finally, that if this continuation arc was pointing to the left, so it would be in both sides of the move and, again, the two final braids would agree. \square

We shall now check the virtual isotopy moves. Indeed we have:

Lemma 8. *Virtual diagrams that differ by virtual isotopy moves correspond to virtual braids that differ by braid isotopy, L_v -moves and real conjugation.*

Proof. We only have to check the moves that involve up-arcs, as the others follow by virtual braid isotopy. We discuss first the RII moves. From all cases of RII moves (with virtual/real crossings, different orientations) the vertical ones checked in Figs. 28 and 29 are the most interesting ones. For example, an RII move placed horizontally follows from two swing moves and a vertical RII move. In Fig. 28 we check a reverse real RII move. Note here that a threaded L_v -move is involved. If this were an over-threaded L_v -move, we could apply Lemma 4 in order to use only an under-threaded one.

In Fig. 29 we check a real RII move with two up-arcs. Here it is real conjugation that will play the main role. Again, the braiding has been done for the rest of the diagram, and parts of it are indicated in grey. The braiding algorithm ensures that there are no other real crossings in the final braid lying in

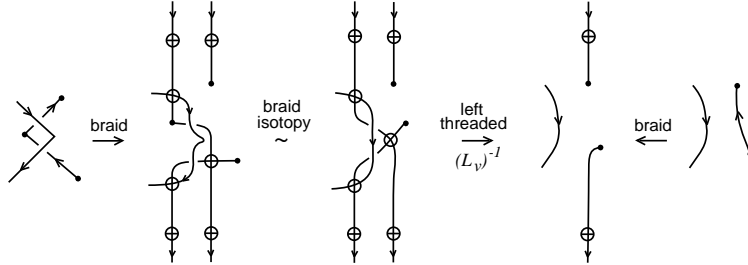


Figure 28: The first case of an RII move.

the narrow vertical zone, which is created after the completion of the braiding. So, apart from the real crossings indicated, all other crossings in this vertical zone will be virtual (drawn in grey), created by the new braid strands. As in the proof of Lemma 3, this means that the old strands act as channels for the real crossings to reach the top and the bottom of the braid, hence to be available for conjugation.

We shall now check the RIII type moves. These include the classical RIII moves, the virtual RIII moves and the special detour moves. Consider an RIII move with one up-arc and two down-arcs. Using a well-known trick we can perform the move using RII moves (which are already checked) and an RIII move with three down-arcs. See Fig. 30. The same trick applies to virtual and special detour moves, but in some cases of special detour moves we may have to also use the swing moves. In Fig. 31 we demonstrate the most interesting case. Finally, if an RIII type move involves two up-arcs and one down-arc we apply the same trick to reduce to the case of one up-arc. Similarly, an RIII type move with three up-arcs reduces to the previous cases. \square

We shall finally check the RI type moves. Virtual RI moves on the diagram clearly give rise to vL_v -moves on the braid level. Braiding a real RI move with downward orientations on the crossing will clearly give rise to a right real L_v -move, if the kink faces the right, and to a left real L_v -move, if the kink faces the left. Braiding a real RI move with upward orientations on the crossing will give rise to a right real L_v -move, if the kink faces the left (see Fig. 32), and to a left real L_v -move, if the kink faces the right. \square

Lemma 9 below completes the proof that all L_v -moves follow from L -equivalence moves.

Lemma 9. *A left real L_v -move can be performed by a sequence of L -equivalence moves. Consequently, an RI move giving rise to a left real L_v -move corresponds to a sequence of L -equivalence moves.*

Proof. For the proof we employ the Whitney trick (compare [15]). In Fig. 33 we start with a virtual diagram K_1 , which is almost the closure of a braid B ,

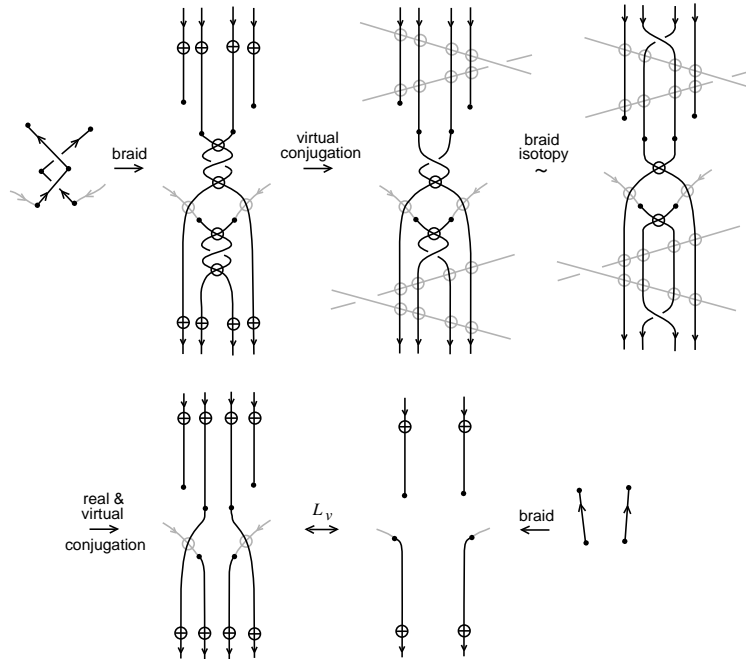


Figure 29: The second case of an RII move.

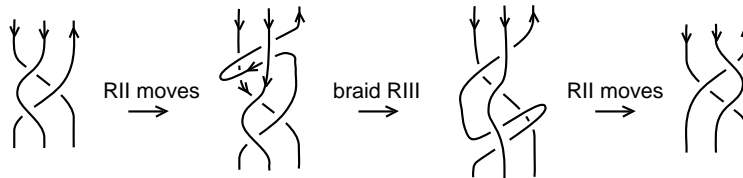


Figure 30: A real RIII move with one up-arc.

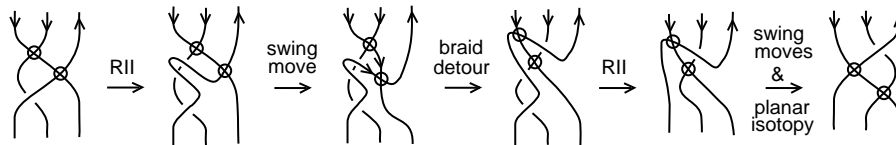


Figure 31: A special detour move with one up-arc.

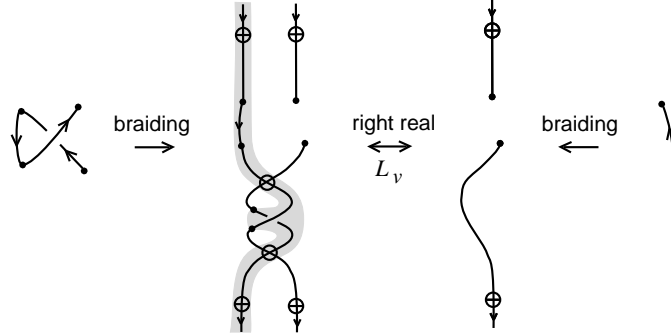


Figure 32: An RI move giving rise to a right real L_v -move.

except that it contains a kink with a real crossing introduced in B . So, K_1 opens to a virtual braid B_1 , which contains a left real L_v -move. On K_1 we introduce a second kink and we perform a sequence of isotopy moves that undo the kink we started with. At the same time we register at each step the difference that every isotopy move makes on the braid level. The final diagram K_9 is, then, the closure of the starting braid B . So, we went from B_1 , containing a left real L_v -move, to B with the L_v -move removed, via a sequence of L -equivalent braids (Definition 6). \square

By Lemmas 1, 2, 4, 7, 8, 9 and by Corollaries 2 and 3 the proof of Theorem 2 is now concluded. \square

Remark 5. As far as the proof of Theorem 2 is concerned, the reverse real RII isotopy moves are the only cases where the threaded L_v -moves appear on the braid level. Moreover, real conjugation is needed in the proof of Lemmas 3 and 4 and in a real RII move with two up-arcs. Finally, the real L_v -moves appear only in the real RI cases.

Conjecture 1. *Real conjugation is not a consequence of the L_v -moves.* In other words, it should be possible to construct a virtual braid invariant that will not distinguish L_v -move equivalent virtual braids, but will distinguish virtual braids that differ by real conjugation. As the simplest possible puzzle, try to show that there is no sequence of L_v -moves connecting the pair of equivalent braids shown in Fig. 34.

5 Algebraic Markov Equivalence for Virtual Braids

In this section we reformulate and sharpen the statement of Theorem 2 by giving an equivalent list of local algebraic moves in the virtual braid groups. More precisely, let VB_n denote the virtual braid group on n strands and let σ_i, v_i be

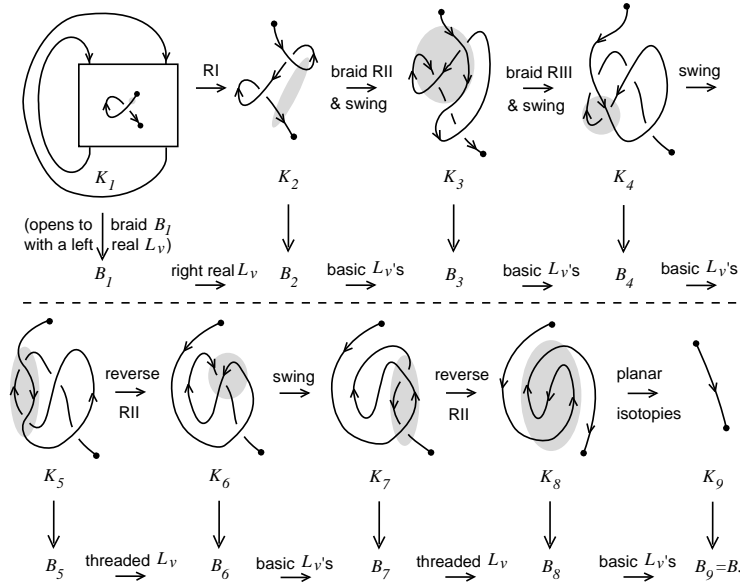


Figure 33: The Whitney trick for the left real L_v -moves.

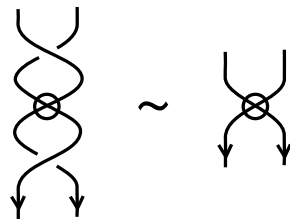


Figure 34: The simplest pair of real conjugates.

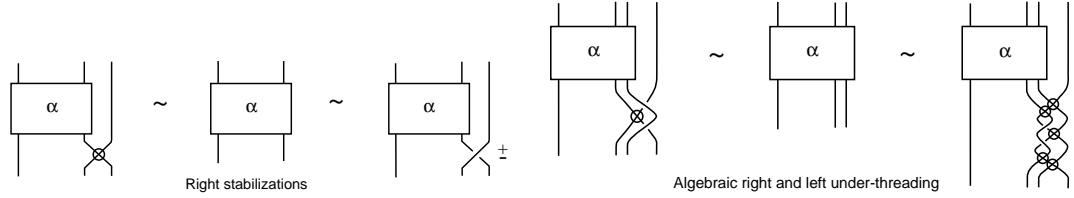


Figure 35: The moves (ii)–(iv) of Theorem 3.

its generating classical and virtual crossings. The σ_i 's satisfy the relations of the classical braid group and the v_i 's satisfy the relations of the permutation group. The characteristic relation in VB_n is the *special detour move* relating both:

$$v_i \sigma_{i+1} v_i = v_{i+1} \sigma_i v_{i+1}.$$

The group VB_n embeds naturally into VB_{n+1} by adding one identity strand at the right of the braid. So, it makes sense to define $VB_\infty := \bigcup_{n=1}^\infty VB_n$, the disjoint union of all virtual braid groups. We can now state our result.

Theorem 3 (Algebraic Markov theorem for virtuals). *Two oriented virtual links are isotopic if and only if any two corresponding virtual braids differ by a finite sequence of braid relations in VB_∞ and the following moves or their inverses:*

- (i) *Virtual and real conjugation:* $v_i \alpha v_i \sim \alpha \sim \sigma_i^{-1} \alpha \sigma_i.$
- (ii) *Right virtual and real stabilization:* $\alpha v_n \sim \alpha \sim \alpha \sigma_n^{\pm 1}.$
- (iii) *Algebraic right under-threading:* $\alpha \sim \alpha \sigma_n^{-1} v_{n-1} \sigma_n^{+1}.$
- (iv) *Algebraic left under-threading:* $\alpha \sim \alpha v_n v_{n-1} \sigma_{n-1}^{+1} v_n \sigma_{n-1}^{-1} v_{n-1} v_n,$

where $\alpha, v_i, \sigma_i \in VB_n$ and $v_n, \sigma_n \in VB_{n+1}$ (see Fig. 35).

Remark 6. Given b in VB_n let $i(b)$ denote the element of VB_{n+1} obtained by adding one to the index of every generating element in b (compare [15]). In other words, $i(b)$ is obtained by adding a single identity strand to the left of b . We also regard b as an element of VB_{n+1} by adding a strand on the right, but take this inclusion for granted, with no extra notation. In the above notation, a left under-threaded L_v -move pulled to the bottom left side of the braid will have the algebraic expression: $\alpha \sim i(\alpha) \sigma_1^{\pm 1} v_2 \sigma_1^{\mp 1}$ (see Fig. 36).

Proof of Theorem 3. The algebraic moves of Theorem 3 follow immediately from the moves of Theorem 2 by braid detouring to the right and by conjugation in VB_∞ . For example, in Fig. 37 we illustrate how to bring a right real L_v -move

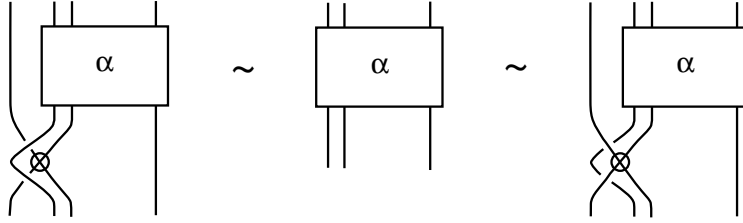


Figure 36: **Bottom left under/over threadings:** $\alpha \sim i(\alpha)\sigma_1^{\pm 1}v_2\sigma_1^{\mp 1}$.

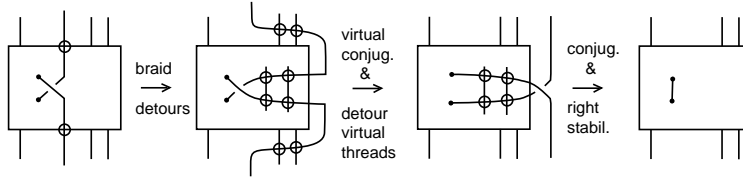


Figure 37: **Right real L_v -move derived from right stabilization.**

to the right end of the braid. In order to derive the algebraic left under-threaded moves: we first bring a left under-threaded L_v -move to the bottom left of the braid by conjugation, and then we braid detour to the right and apply virtual conjugation. \square

Remark 7. By the braid conjugation, moves (ii), (iii), (iv) of Theorem 3 could be equally given with the local algebraic part in between two braids. For example:

$$\alpha\beta \sim \alpha\sigma_n^{-1}v_{n-1}\sigma_n^{+1}\beta.$$

Finally, we should point out that the proof in Fig. 37 can be also adapted to the case of allowed classical L -moves, namely pulling to the right or left, depending on which side is free of virtual crossings. Here, the conjugation for pulling aside is real and agrees with the type (over/under) of the classical L -move. Once out of the braid box, we have a real stabilization move.

6 Kamada’s Markov Theorem for Virtual Braids

In this section we present Kamada’s Markov theorem for virtual braids [15] and we show that our Theorem 3 is equivalent to the theorem of Kamada. With the inclusion of braids of Remark 6, Kamada proved the following:

Theorem (Kamada [15]) *Two virtual braids b and b' have isotopic closures if and only if they are related to one another through a finite sequence of braid relations in VB_∞ and the following moves:*

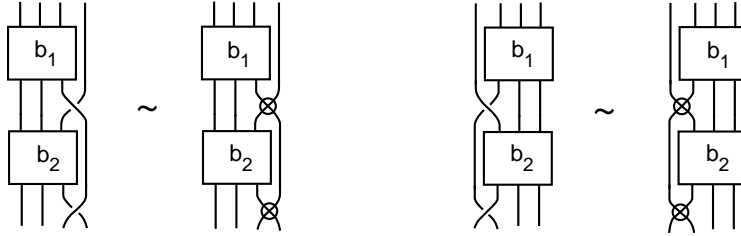


Figure 38: The right and left exchange moves of Kamada.

1. conjugation if b' is the conjugation of $b \in VB_n$ by an element of VB_n .
2. right stabilization move if b' is $b\sigma_n$ or $b\sigma_n^{-1}$ or $bv_n \in VB_{n+1}$, for $b \in VB_n$.
3. right exchange move if they belong to one of the following patterns, for $b_1, b_2 \in VB_n$

$$\{b_1\sigma_n^{-1}b_2\sigma_n, b_1v_nb_2v_n\}.$$

4. left exchange move if they belong to one of the following patterns, for $b_1, b_2 \in VB_n$

$$\{i(b_1)\sigma_1^{-1}i(b_2)\sigma_1, i(b_1)v_1i(b_2)v_1\}.$$

In Fig. 38 we illustrate the braids for the right and left exchange moves. It is clear from the figure that the corresponding braids for these moves have equivalent closures.

Proposition 1. *The moves of the Kamada theorem follow from the moves of Theorem 3. Conversely, the moves of Theorem 3 can be realized via the moves of Kamada.*

Proof. The first two moves coincide with moves (i) and (ii) of Theorem 3. Further, Kamada’s exchange moves are special cases of in-box exchange moves (recall Definition 7), so, by Lemma 3 and by Theorems 2 and 3, they follow from the moves of Theorem 3.

Consider now an algebraic right under-threaded move. The one side of the move is a special case of one side of the exchange move, where the second braid box contains only the virtual crossing v_{n-1} . Perform the exchange move to change the thread to a virtual one. Braid detour it away and apply the right virtual stabilization. This brings us to the other side of the threaded move. Finally, note that Kamada’s left exchange move is equivalent, up to conjugation, to a similar left exchange move with the opposite crossings. Let us call that one an ‘under left exchange move’. For realizing an algebraic left under-threaded move: conjugate it first to the bottom left of the braid (as in Fig. 36) and realize this move via an under left exchange move. Then conjugate the result back to the bottom right of the braid. \square

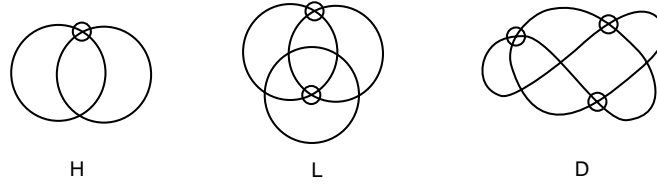


Figure 39: Examples of flat knots and links.

7 The Markov Theorem for Flat Virtuals and Welded Links

In this section we give the analogs of Theorems 2 and 3 for flat virtuals, welded links and virtual unrestricted links. Each category is interesting on its own right and has been studied by various authors. In [21] we gave reduced presentations for the corresponding braid groups.

7.1 Flat Virtuals

Every classical knot or link diagram can be regarded as an immersion of circles in the plane with extra under/over structure at the double points. If we take the diagram without this extra structure, it is the shadow of some link in three dimensional space, but the weaving of that link is not specified. We call these shadow crossings *flat crossings*. Clearly, if one is allowed to apply the Reidemeister moves to a shadow diagram (without regard to the types of crossings) then the diagram can be reduced to a disjoint union of circles. This reduction is no longer true in the presence of virtual crossings.

More precisely, let a *flat virtual diagram* be a diagram with flat crossings and virtual crossings. Two flat virtual diagrams are *equivalent* if there is a sequence of *flat virtual Reidemeister moves* taking one to the other. These are moves as shown in Fig. 2, but with flat crossings in place of classical crossings. Note that in the category of flat virtuals there is only one forbidden move. Detour moves as in Fig. 2(C) are available only for virtual crossings with respect to flat crossings and not the other way around. The study of flat virtual knots and links was initiated in [18]. The category of flat virtual knots is identical in structure to what are called *virtual strings* by Turaev in [36].

Fig. 39 illustrates flat virtual links H and L and a flat virtual knot D . The link H cannot be undone in the flat category because it has an odd number of virtual crossings between its two components and each flat virtual Reidemeister move preserves the parity of the number of virtual crossings between components. The diagram D is shown to be a non-trivial flat virtual knot using the filamentation invariant, see [11]. The diagram L is also a non-trivial flat diagram. Note that it comes apart at once if we allow the forbidden move.

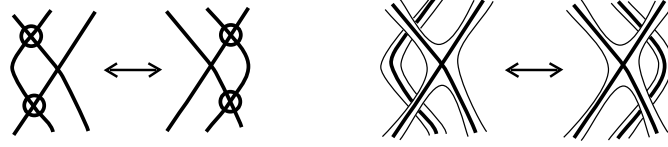


Figure 40: Flat version of the detour move.

Just as virtual knots and links can be interpreted via stabilized embeddings of curves in thickened surfaces, flat virtuals can be interpreted as stabilized immersions of curves in surfaces (no thickening required). See [13] for applications of this point of view. Similarly, flat virtual links and braids have ribbon surface interpretations. In Fig. 40 we illustrate the flat special detour move and its local ribbon surface embedding. Note the stark difference here between the virtual crossing structure and the immersion structure of the flat crossings.

We shall say that a virtual diagram *overlies* a flat diagram if the virtual diagram is obtained from the flat diagram by choosing a crossing type for each flat crossing in the virtual diagram. To each virtual diagram K there is an associated flat diagram $F(K)$ that is obtained by forgetting the extra structure at the classical crossings in K . Note that if K is equivalent to K' as virtual diagrams, then $F(K)$ is equivalent to $F(K')$ as flat virtual diagrams. Thus, if we can show that $F(K)$ is not reducible to a disjoint union of circles, then it will follow that K is a non-trivial virtual link.

The flat virtual braids were introduced in [19]. As with the virtual braids, the set of flat virtual braids on n strands forms a group, the *flat virtual braid group*, denoted FV_n . The generators of FV_n are the virtual crossings v_i and the flat crossings c_i , such that $c_i^2 = 1$. Both, flat crossings and virtual crossings represent geometrically the generators of the symmetric group S_n . But the *mixed relation* between them:

$$v_i c_{i+1} v_i = v_{i+1} c_i v_{i+1}$$

is not symmetric (see Fig. 40). FV_n is a quotient of the virtual braid group VB_n modulo the relations $\sigma_i^2 = 1$ for all i . Thus, FV_n is the free product of two copies of S_n , modulo the set of mixed relations. Note that $FV_2 = S_2 * S_2$ (no extra relations), and it is infinite.

From the above, the flat virtual braids are the appropriate theory of braids for the category of virtual strings. Every virtual string is the closure of a flat virtual braid. In order to obtain a Markov theorem for flat virtual braids, we only need to forget, in our study of virtuals and the definitions of the virtual L -moves, the distinction between over and under crossings. The presence of the flat forbidden move gives rise to the *flat threaded L_v -moves*, left and right, the analogues of the threaded L_v -moves. Figs. 11, 12 and 13 provide illustrations,

if we substitute the real crossings by flat ones. Thus, we have the following results.

Theorem 4 (L -move Markov theorem for flat virtuals). *Two oriented flat virtual links are isotopic if and only if any two corresponding flat virtual braids differ by flat virtual braid isotopy and a finite sequence of the following moves or their inverses:*

- (i) Flat conjugation.
- (ii) Right virtual L_v -moves.
- (iii) Right flat L_v -moves.
- (iv) Right and left flat threaded L_v -moves.

Theorem 5 (Algebraic Markov theorem for flat virtuals). *Two oriented flat virtual links are isotopic if and only if any two corresponding flat virtual braids differ by braid relations in FV_∞ and a finite sequence of the following moves or their inverses:*

- (i) Virtual and flat conjugation: $v_i \alpha v_i \sim \alpha \sim c_i \alpha c_i$.
- (ii) Right virtual and flat stabilization: $\alpha v_n \sim \alpha \sim \alpha c_n$.
- (iii) Algebraic right flat threading: $\alpha \sim \alpha c_n v_{n-1} c_n$.
- (iv) Algebraic left flat threading: $\alpha \sim \alpha v_n v_{n-1} c_{n-1} v_n c_{n-1} v_{n-1} v_n$.

where $\alpha, v_i, c_i \in FV_n$ and $v_n, c_n \in FV_{n+1}$. (Fig. 35 provides illustrations, substituting the real crossings by flat ones).

7.2 Welded Links and Unrestricted Virtuals

Welded braids were introduced in [8]. They satisfy the same isotopy relations as the virtuals, but for welded braids one of the two forbidden moves of Fig. 4 is allowed, the move F_1 , which contains an *over arc* and one virtual crossing. One can consider welded knots and links as closures of welded braids. The move F_1 can be regarded as a way of detouring sequences of classical crossings *over* welded crossings. The explanation for the choice of moves lies in the fact that the move F_1 preserves the combinatorial fundamental group. This is not true for the other forbidden move F_2 . The *welded braid group* on n strands, WB_n , is a quotient of the virtual braid group, so it can be presented with the same generators and relations as VB_n , but with the extra relations:

$$v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1} \quad (F_1).$$

In order to obtain a Markov type theorem for welded braids, we only need to consider in our study of virtuals the effect of the move F_1 . The presence of this move makes redundant the under-threaded L_v -moves since, by the move

F_1 , the thread can be pulled away, reducing the move to a basic vL_v -move. Thus, threading disappears from the theory of welded braids and we have the following results (compare [15]).

Theorem 6 (L-move Markov theorem for welded knots). *Two oriented welded links are isotopic if and only if any two corresponding welded braids differ by welded braid isotopy and a finite sequence of the following moves or their inverses:*

- (i) *Real conjugation.*
- (ii) *Right virtual L_v -moves.*
- (iii) *Right real L_v -moves.*

Theorem 7 (Algebraic Markov theorem for welded knots). *Two oriented welded links are isotopic if and only if any two corresponding virtual braids differ by braid relations in WB_∞ and a finite sequence of the following moves or their inverses:*

- (i) *Virtual and real conjugation:* $v_i \alpha v_i \sim \alpha \sim \sigma_i^{-1} \alpha \sigma_i.$
- (ii) *Right virtual and real stabilization:* $\alpha v_n \sim \alpha \sim \alpha \sigma_n^{\pm 1},$

where $\alpha, v_i, \sigma_i \in WB_n$ and $v_n, \sigma_n \in WB_{n+1}$ (recall Fig. 35 for illustrations).

This statement of the Markov theorem for welded braids is equivalent to that of Kamada [15].

Finally, another quotient of the virtual braid group (and of the welded braid group) is obtained by adding both types of forbidden moves. We call this the *unrestricted virtual braid group*, denoted UB_n . It is known that any classical knot can be unknotted in the virtual category if we allow both forbidden moves [17, 31]. Nevertheless, linking phenomena still remain. The unrestricted virtual braid group itself is non trivial, deserving further study. For a presentation of UB_n we just add to the presentation of VB_n both types of forbidden moves:

$$v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1} \quad (F_1) \quad \text{and} \quad \sigma_i \sigma_{i+1} v_i = v_{i+1} \sigma_i \sigma_{i+1} \quad (F_2).$$

Then we have the following:

Theorem 8 (Algebraic Markov theorem for unrestricted virtuals). *Two oriented unrestricted virtual links are isotopic if and only if any two corresponding unrestricted virtual braids differ by braid relations in UB_∞ and a finite sequence of the following moves or their inverses:*

- (i) *Virtual and real conjugation:* $v_i \alpha v_i \sim \alpha \sim \sigma_i^{-1} \alpha \sigma_i.$
- (ii) *Right virtual and real stabilization:* $\alpha v_n \sim \alpha \sim \alpha \sigma_n^{\pm 1},$

where $\alpha, v_i, \sigma_i \in UB_n$ and $v_n, \sigma_n \in UB_{n+1}$.

Note that the moves of the equivalence relations in Theorems 7 and 8 are apparently the same. The difference in the theory lies in the different structures of the corresponding braid groups.

8 On Virtual R -matrices and Virtual Hecke Algebras

In this section we illustrate relations on an R -matrix solution to the Yang-Baxter equation that would allow an analog of the Markov trace construction to be made for virtual braids. Such a construction leads to invariants of virtual knots and links, yielding valuable information about the virtual category. In Fig. 41 we illustrate the apparatus and relations that are needed to construct a Markov trace on braids from an R -matrix in the classical case.

The illustration uses diagrammatic matrix notation. In this notation a matrix or tensor is represented by a box or otherwise delineated polygon in the plane with strands emanating from the box, indicating the indices of the matrix. When a line from one diagrammatic matrix is tied with a line from another, we see an internal edge in the graphical structure and this is interpreted as a shared index in the matrix interpretation. Thus, at the matrix level one sums over all possible indices that label an internal edge, and one takes the products of all the matrix entries concerned. This is an exact generalization of the formula for matrix multiplication

$$(MN)_{ab} = \sum_i M_{ai} N_{ib}$$

where summation is over all indices i relevant to this matrix product.

One can conceptualize diagrammatic matrices by regarding the diagrams as morphisms in a graphical category, and the interpretation as matrix multiplication as a functor to a linear algebraic category. The same remarks apply to the well-known Einstein summation convention where we write

$$M_{ai} N_{ib}$$

and interpret the repeated index as a summation over all values for i . Here the algebraic notation $M_{ai} N_{ib}$ is in an abstract tensor category of indexed algebra with rules for handling repeated indices. For example, $M_{ai} N_{ib} = M_{aj} N_{jb}$ so long as j is also repeated and j denotes a letter distinct from a and b . The interpretation as summation takes the abstract tensor category to a linear algebra category. The diagrams are a generalization of the abstract tensor category. We use this diagrammatic matrix algebra in our illustrations to show the translation from the category of link diagrams and virtual link diagrams to the matrix algebraic formulas that can capture an invariant of virtual knots and links via the Markov theorem. For example, see Fig. 41.

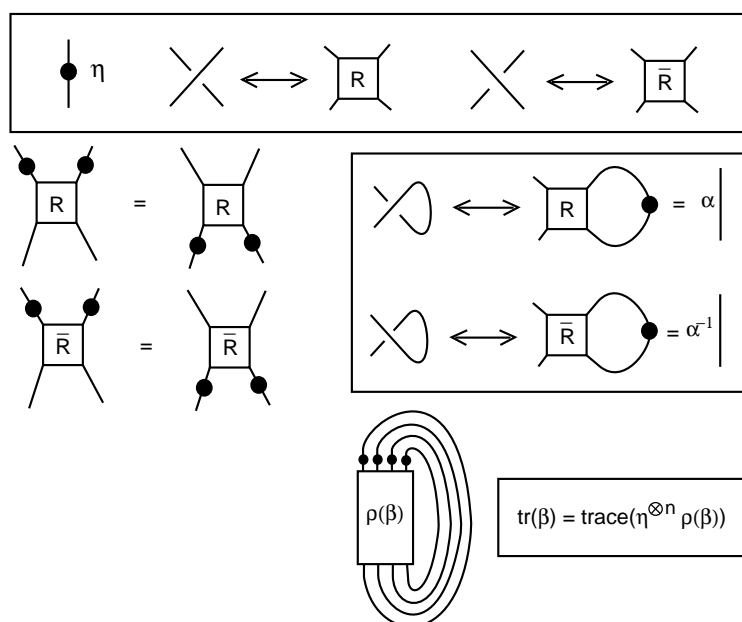


Figure 41: R -matrix relations.

The first diagram at the upper left denotes a matrix η_j^i where the indices are designated by the strands emanating from the black disk that is the body of the diagrammatic version of η . Crossings are represented by the matrices $R = (R_{kl}^{ij})$ and $\bar{R} = R^{-1}$. These matrices must satisfy a braiding relation that corresponds to the third Reidemeister move. At the matrix level this relation is called the Yang-Baxter Equation. We have not illustrated this relation. The virtual crossings are shown in Fig. 42. They are represented by a matrix V that must also satisfy the Yang-Baxter equation and the detour relations that generate the virtual braid group. These matrices then generate a tensor representation of the virtual braid group where a generator acting on the i -th and $(i + 1)$ -st strands receives an R, \bar{R} or V , according as it is a classical or virtual crossing, and all the other strands receive an identity matrix. Given a virtual braid β , let $\rho(\beta)$ denote this representation applied to β .

Now return to Fig. 41. Note that we define a trace function on braids by the formula

$$tr(\beta) = trace(\eta^{\otimes n} \rho(\beta)).$$

Here *trace* denotes the usual trace of a matrix. This trace formula is indicated diagrammatically by the figure in the box to its immediate left. In order for $tr(\beta)$ to be constructed (after normalization) as a virtual link invariant, we need:

1. $tr(\beta\gamma) = tr(\gamma\beta)$, for any braids β and γ ,
2. $tr(\beta)$ should either be invariant or it should multiply by a constant under stabilization moves and under-threaded moves.

In Fig. 41 we have indicated $tr(\beta)$ to multiply by α under right positive classical stabilization and by α^{-1} under right negative classical stabilization. In Fig. 42 we have indicated right virtual stabilization invariance. In Fig. 42 we also illustrate the diagrammatics of a right under-threaded move. Note that these stabilization equations all involve the matrix η . Appropriate choices of the solutions to the Yang-Baxter equation and the matrix η can, in principle, lead to invariants of both classical and virtual knots and links. One obtains a normalized invariant $Invar(b)$ by the formula

$$Invar(b) = \alpha^{-w(b)} tr(b)$$

where $w(b)$ is the sum of the signs of the exponents of the classical braid generators in an expression for the braid b .

One case is worth mentioning here explicitly. Suppose that η and R yield an invariant of classical braids (of which there are many, including the Jones polynomial and specializations of the homflypt polynomial). Then we can take V (as a linear mapping) to be the permutation $V(x \otimes y) = y \otimes x$. Under these conditions $tr(b)$ will satisfy classical stabilization, but will not necessarily satisfy virtual stabilization. We call such invariants *virtual rotational invariants*. They are interesting in their own right. It is a subtle matter to obtain full virtual invariants, but there are examples, including the Jones polynomial itself.

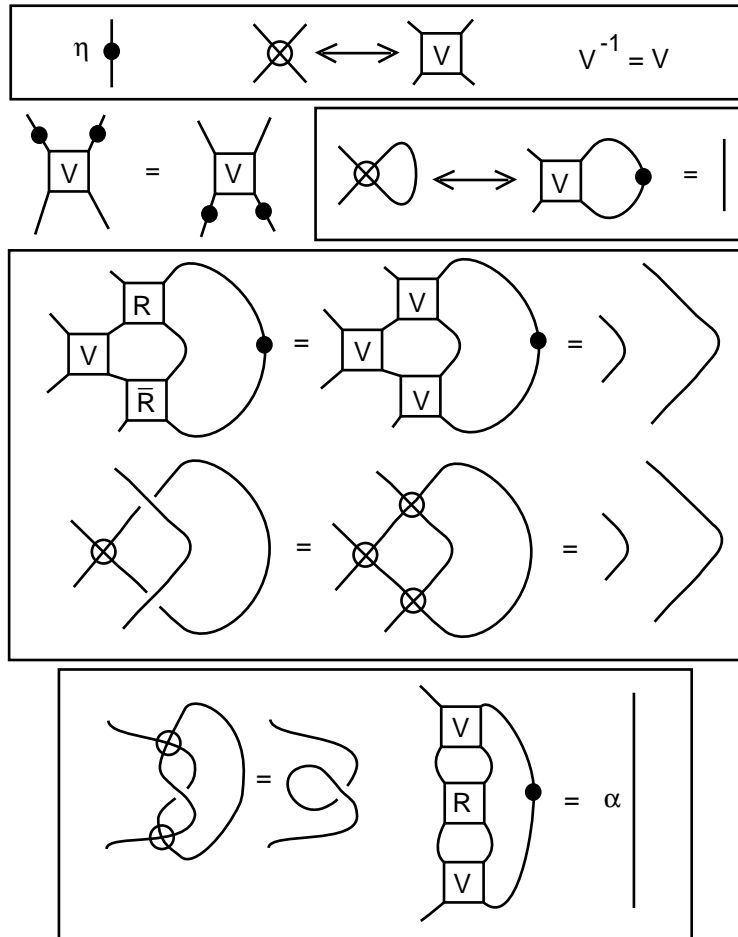


Figure 42: Virtual R -matrix relations.

Theorem 3 opens up yet another possibility to construct invariants of virtual links using algebraic means. Namely, to study quotients of the virtual braid group algebra and try to construct on them linear Markov-type traces. Then, to apply appropriate normalizations yielding virtual link invariants. Taking the lead of Jones's construction [12] of the homflypt (2-variable Jones) polynomial we define $\mathcal{VH}_n(q)$, the *virtual Hecke algebra* as the quotient of the virtual braid group algebra $\mathbb{Z}[q^{\pm 1}]VB_n$ by factoring out the quadratic relations:

$$\sigma_i^2 = (q - 1)\sigma_i + q.$$

Let $g_1, \dots, g_n, v_1, \dots, v_n$ be the generators of $\mathcal{VH}_{n+1}(q)$. A *virtual Markov trace* is defined to be a linear function tr on $\bigcup_{n=1}^{\infty} \mathcal{VH}_n(q)$ which supports the real and virtual Markov properties. More precisely, we require the trace tr to satisfy the rules:

- 1) $tr(ab) = tr(ba)$,
- 2) $tr(1) = 1$ for all $\mathcal{VH}_n(q)$,
- 3) $tr(ag_n) = z tr(a)$,
- 4) $tr(av_n) = s tr(a)$,
- 5) $tr(ag_n^{-1}v_{n-1}g_n^{+1}) = r tr(a)$,
- 6) $tr(av_nv_{n-1}g_{n-1}^{+1}v_n g_{n-1}^{-1}v_{n-1}v_n) = k tr(a)$,

for $a, b \in \mathcal{VH}_n(q)$ and z, s, r, k independent variables in $\mathbb{Z}[q^{\pm 1}]$. Finally, we normalize tr appropriately in order to obtain an invariant of virtual links.

We will pursue these matters of R -matrix invariants of virtual braids and virtual Hecke algebras in a subsequent paper.

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